Monetary Economics

Chapter 2: Optimal Monetary Policy

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October – November 2024

Objective of the chapter I

- This chapter studies **optimal conventional monetary policy** in the basic NK model, i.e. how the short-term nominal interest rate should be set in order to maximize RH's utility in this model, both
    - at the steady state ($\equiv$ constant equilibrium in the absence of shocks),
    - in response to shocks.

- By comparing the market equilibrium to the social-planner allocation, it identifies the **two distortions** that prevent the First Welfare Theorem from being applicable and give a role to monetary policy (MP).

- It shows that the objective of MP should be to stabilize both
    - a specific output gap,
    - the inflation rate,

  thus providing a justification for the "**flexible inflation-targeting strategies**" adopted by many central banks (CBs).

Objective of the chapter II

- Lastly, the chapter studies both
  - optimal MP under **discretion** (or "optimal time-consistent MP") $\equiv$ MP conducted when, at each date t, CB chooses i_t to maximize $\mathcal{U}_t$,
  - optimal MP under **commitment** $\equiv$ MP conducted when, at date 0, CB chooses the state-contingent path $(i_t)_{t\geq 0}$ to maximize $\mathcal{U}_0$.

- When the two are different, optimal MP under commitment is said to be **time-inconsistent** (Kydland and Prescott, 1977).

- The chapter identifies two sources of time-inconsistency, giving rise to
  - an **inflation bias**,
  - a **stabilization bias**,

  under discretion (relatively to under commitment).

Outline of the chapter

Social-planner allocation I

- Consider a **benevolent social planner** seeking to maximize RH's welfare given technology.

- Given the absence of state variable (such as the capital stock), its optimization problem is **static**: at each date t,

$$\underset{[C_t(i)]_{0\leq i\leq 1},[N_t(i)]_{0\leq i\leq 1}}{Max} U(C_t, N_t)$$

  subject to

$$C_t \equiv \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

$$N_t = \int_0^1 N_t(i)di,$$

$$C_t(i) = A_t N_t(i)^{1-\alpha} \text{ for } i \in [0,1].$$

Social-planner allocation II

- The **optimality conditions** are

$$C_t(i) = C_t \text{ for } i \in [0,1],$$
$$N_t(i) = N_t \text{ for } i \in [0,1],$$
$$-\frac{U_{n,t}}{U_{c,t}} = MPN_t,$$

where $MPN_t \equiv (1-\alpha) A_t N_t^{-\alpha}$ is the average marginal product of labor.

- The **first and second conditions** come from both
  - the strict concavity of C_t in each $C_t(i)$ (when $\varepsilon < +\infty$), which implies a preference for smoothing consumption across differentiated goods,
  - the strict concavity of $C_t(i)$ in $N_t(i)$ (when $\alpha > 0$), which implies that it is optimal to smooth labor services across goods even when $\varepsilon \to +\infty$.

- The **third condition** equalizes the marginal rate of substitution between consumption and work to the corresponding marginal rate of transformation.

Two distortions

- The basic NK model is characterized by **two distortions**:
  1. monopolistic competition,
  2. sticky prices.

- The **first distortion** is effective
  - both at the steady state (unless it is exactly offset by the constant subsidy τ) and in response to shocks,
  - both when prices are sticky ($\theta > 0$) and when they are flexible ($\theta = 0$).

- The **second distortion** is effective
  - only in response to shocks, not at the steady state,
  - only when prices are sticky ($\theta > 0$), not when they are flexible ($\theta = 0$).

First distortion: Monopolistic competition I

- Consider the case in which prices are fully flexible ($\theta = 0$), in order to focus on the **monopolistic-competition distortion**.

- Then, all firms set the same price, equal to

$$P_t = \mathcal{M} \frac{(1-\tau) W_t}{MPN_t},$$

where $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon - 1} > 1$ is the (gross) markup under flexible prices and $\frac{(1-\tau)W_t}{MPN_t}$ is the nominal marginal cost.

First distortion: Monopolistic competition II

- Therefore, using RH's intratemporal optimality condition, we get

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} = \frac{MPN_t}{(1-\tau)\mathcal{M}} \neq MPN_t$$

unless $\tau = \frac{1}{\varepsilon}$ (distortion exactly offset by subsidy).

- In particular, for $\tau = 0$ (no subsidy), we get

$$-\frac{U_{n,t}}{U_{c,t}} < MPN_t,$$

which implies an inefficiently low level of employment and output (given that $-U_{n,t}$ is increasing, and $U_{c,t}$ and MPN_t decreasing, in work hours).

Second distortion: Sticky prices I

- Now suppose that the subsidy exactly offsets the monopolistic-competition distortion ($\tau = \frac{1}{\varepsilon}$), in order to isolate the **sticky-prices distortion**.

- Then, noting M_t the average markup (defined as the ratio of average price to average nominal marginal cost), we have

$$M_t \equiv \frac{P_t}{\frac{(1-\tau_t)W_t}{MPN_t}} = \frac{P_t \mathcal{M} MPN_t}{W_t}$$

and therefore

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} = MPN_t \frac{\mathcal{M}}{M_t} \neq MPN_t$$

since $M_t \neq \mathcal{M}$ generically (because of sticky prices).

- Thus, sticky prices **distort the average price** and imply either too low or too high a level of aggregate employment and output.

Second distortion: Sticky prices II

- Sticky prices also **distort the relative prices** of different goods, making them vary in a way unrelated to changes in preferences or technology.
- This is due to the lack of synchronization in price adjustments: newly reset prices will generically differ from other prices.
- This price dispersion across goods ($P_t(i) \neq P_t(j)$) leads to a dispersion in quantities consumed ($C_t(i) \neq C_t(j)$) and in labor services ($N_t(i) \neq N_t(j)$).
- These dispersions violate the first and second optimality conditions characterizing the social-planner allocation.

Optimal MP in a simple case I

- Consider the following **specific case**:
    - the steady state is efficient: $\tau = \frac{1}{\varepsilon}$,
    - there is no initial price dispersion: $P_{t-1}(i) = P_{t-1}$ for all $i \in [0, 1]$.

- The first assumption **removes the first distortion**.

- Recall from Chapter 1 that

$$p_t^* - p_{t-1} = \mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \left[(1-\beta\theta)\Theta\left(\mu + mc_{t+k}\right) + \pi_{t+k} \right] \right\}.$$

- Consider a MP that, at each date $t+k$, ensures that $(1-\beta\theta)\Theta\left(\mu + mc_{t+k}\right) + \pi_{t+k} = 0$.

- Then $p_t^* = p_{t-1}$ and, hence, $\pi_t = 0$ for each date $t \in \mathbb{N}$.

Optimal MP in a simple case II

- Thus, the aggregate price level is perfectly stabilized and **no relative-price distortions** emerge.
- In addition, this MP implies $\mu + mc_{t+k} = 0$, and hence $w_{t+k} - p_{t+k} - mpn_{t+k} = 0$, so there are **no aggregate-price distortions**.
- Therefore, this MP **removes the second distortion**.
- Therefore, this MP **replicates the social-planner allocation**.
- Therefore, this MP **is optimal**.

Optimal MP in a simple case III

- Since the social-planner allocation in this case is the flexible-price allocation, we have $\widetilde{y}_t = 0$ under this MP: **output is stabilized at its natural (i.e., flexible-price) level**.

- Under this MP, $\pi_t = 0$: **inflation is stabilized at a constant (zero) level**.

- This is because the only way to replicate the (efficient) flexible-price allocation when prices are sticky is by making all firms satisfied with their existing prices (so that **the sticky-price constraint is not binding**).

- Under this MP, $i_t = r_t^n$: **the nominal interest rate tracks the natural rate of interest**.

Optimal MP in a simple case IV

- This result, obtained for technology shocks, extends to all **non-distortive shocks**, like consumption-utility and labor-disutility shocks.

- However, it does not extend to shocks to the elasticity of substitution between differentiated goods (i.e. exogenous stochastic variations in ε), called **cost-push chocks**.

- Nor does it (fully) extend to **non-efficient steady states** ($\tau \neq \frac{1}{\varepsilon}$).

- In the rest of the chapter, we study optimal MP
  - in the presence of **cost-push chocks**,
  - when the steady state is **inefficient**.

Introducing cost-push shocks I

- We now introduce **cost-push shocks** into the model: the elasticity of substitution between goods is now ε_t and fluctuates exogenously around the steady-state value ε (while remaining above 1).

- The **IS equation**

$$y_t = \mathbb{E}_t \{y_{t+1}\} - \frac{1}{\sigma} (i_t - \mathbb{E}_t \{\pi_{t+1}\})$$

  is **unchanged**, as it comes from the Euler equation and the goods-market-clearing condition (none of which is affected by cost-push shocks).

- We admit that the **Phillips curve**

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa (y_t - y_t^n)$$

  is also **unchanged**.

Introducing cost-push shocks II

- **Intuition** for the unchanged Phillips curve:
  - the only possible change would be that the exog. term is no longer y_t^n;
  - however, for $\mathbb{E}_t\{\pi_{t+1}\} = 0$, what matters for the average resetting firm is still how y_t compares to y_t^n;
  - for $y_t > y_t^n$ (resp. $y_t < y_t^n$), the price-stickiness constraint is binding for the average non-resetting firm, which would like to raise (resp. cut) its price; so, the average resetting firm raises (resp. cuts) its price;
  - for $y_t = y_t^n$, the price-stickiness constraint is not binding for the average non-resetting firm, which would not have changed its price if allowed to; so, the average resetting firm does not change its price;
  - as price stickiness vanishes ($\theta \to 0$ and hence $\kappa \to +\infty$), the Phillips curve converges to $y_t = y_t^n$.

Introducing cost-push shocks III

- However, the **natural level of output y_t^n changes**, as under flexible prices the output level now depends on cost-push shocks: instead of

$$y_t^n = \frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha}\left[\log\left(\frac{1-\alpha}{1-\tau}\right)+\frac{1+\varphi}{1-\alpha}a_t-\mu\right],$$

where $\mu \equiv \log[\varepsilon/(\varepsilon-1)]$, we now have

$$y_t^n = \frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha}\left[\log\left(\frac{1-\alpha}{1-\tau}\right)+\frac{1+\varphi}{1-\alpha}a_t-\mu_t\right],$$

where $\mu_t \equiv \log[\varepsilon_t/(\varepsilon_t-1)]$.

- We have $\varepsilon_t \nearrow \Rightarrow \mu_t \searrow \Rightarrow y_t^n \nearrow$: an increase in the elasticity of substitution between goods reduces firms' market power and raises flexible-price output.

New variables I

- We want to determine optimal MP in the general case, i.e. in the presence of cost-push shocks and when the steady state is inefficient.

- To do so, we derive the second-order approximation of RH's utility around the zero-inflation-rate steady state (ZIRSS).

- Before deriving this approximation, it is useful to define a few new variables.

- Let $\overline{y}_t^n$ denote the equilibrium output level under flexible prices ($\theta = 0$) and a constant elasticity of substitution between goods ($\varepsilon_t = \varepsilon$):

$$\overline{y}_t^n = \frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha}\left[\log\left(\frac{1-\alpha}{1-\tau}\right)+\frac{1+\varphi}{1-\alpha}a_t-\mu\right],$$

and let $x_t \equiv y_t - \overline{y}_t^n$ denote an **alternative output gap** ($x_t \neq \tilde{y}_t$).

New variables II

- Let y_t^e denote the **efficient level of output** at t, defined as the level of output that would be chosen by a benevolent social planner at t.
- The first welfare theorem implies that $y_t = y_t^e$ under
  - flexible prices ($\theta = 0$),
  - perfect competition ($\mu_t = 0$),
  - no employment subsidy ($\tau = 0$).
- Therefore, the efficient level of output is

$$y_t^e = \frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha}\left[\log(1-\alpha) + \frac{1+\varphi}{1-\alpha}a_t\right].$$

- Finally, let x^* denote the **degree of steady-state inefficiency**:

$$x^* \equiv y_t^e - \overline{y}_t^n = \frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha}\left[\log\left(1-\tau\right)+\mu\right],$$

with $x^* = 0$ if $\tau = \frac{1}{\varepsilon}$ and $x^* > 0$ if $\tau < \frac{1}{\varepsilon}$ (we rule out the case $\tau > \frac{1}{\varepsilon}$).

Welfare-loss function I

- As shown in the Appendix, maximizing RH's utility is equivalent, at the second order, to minimizing the **welfare-loss function**

$$L_0 \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\pi_t^2 + \lambda (x_t - x^*)^2 \right] \right\}, \quad \text{where} \quad \lambda \equiv \frac{\kappa}{\varepsilon}.$$

- **L_t involves π_t** because of price stickiness:
  - every variation in the general level of prices (i.e. every deviation of π_t from zero) implies a price dispersion,
  - this price dispersion is sub-optimal given the strict concavity of C_t in each $C_t(i)$ (when $\varepsilon < +\infty$) and of each $C_t(i)$ in $N_t(i)$ (when $\alpha > 0$).

- **L_t involves $x_t - x^*$** $= y_t - y_t^e$ simply because any deviation of y_t from y_t^e is sub-optimal. Note that the efficient output level y_t^e
  - does not depend on cost-push shocks (i.e. market-power variations),
  - "compensates" for steady-state inefficiency.

Welfare-loss function II

- **The relative weight λ** of the x_t-stabilization objective **is decreasing in**
  - **the degree of price stickiness θ**: the stickier the prices, the longer the effect of a given change in inflation on price dispersion...
  - **the elasticity of intertemporal substitution $\frac{1}{\sigma}$**: the larger this elasticity, the lower the effect of a given output-gap variance on consumption utility...
  - **the Frisch elasticity of labor $\frac{1}{\varphi}$**: the larger this elasticity, the lower the effect of a given output-gap variance on labor disutility...
  - **the elasticity of substitution between differentiated goods ε**: the larger this elasticity, the larger the effect of a given price dispersion on output dispersion and labor dispersion...

...and therefore the more important the π_t-stabilization objective relatively to the x_t-stabilization objective.

Welfare-relevant vs. usual inflation/output-gap measures

- Woodford (2003a, C6) considers an extended NK model in which the degree of price stickiness θ varies across the differentiated goods.
- He shows that the **welfare-relevant inflation measure** (i.e. the one appearing in the welfare-loss function) assigns to each sector **a weight that is an increasing function of its degree of price stickiness**.
- This measure is closer to core-inflation measures (i.e. inflation measures excluding volatile prices) than to the usual headline-inflation measure that most central banks have chosen to (try to) stabilize.
- The **welfare-relevant output-gap measure** (i.e. the one appearing in the welfare-loss function) is the difference between the actual output level y_t and the flexible-price cost-push-shocks-free output level $\overline{y}_t^n$.
- This measure is very different from **usual output-gap measures** (i.e. detrended-output measures), as $\overline{y}_t^n$ is much less smooth than the output trend.

State-contingent path under optimal MP

- We now consider the **state-contingent path** (i.e., the value taken by current endogenous variables as a function of current and past exogenous shocks) followed by the economy **under optimal MP**.

- We are interested in the **first-order approximation of this path** around the zero-inflation-rate steady state (ZIRSS), which we note $\mathcal{P}^{\star}$.

- We have just obtained, in the previous section, the
  - first-order approximation of the structural equations around the ZIRSS,
  - second-order approximation of RH's utility function around the ZIRSS.

- The path that maximizes the latter subject to the former (**linear-quadratic optimization problem**) coincides with $\mathcal{P}^{\star}$ if x^* is sufficiently small (i.e. of the same order of magnitude as shocks).

Rewriting the IS equation and Phillips curve

- Since the **loss function** is expressed in terms of $x_t \equiv y_t - \overline{y}_t^n$, we first rewrite the **IS equation** and **Phillips curve** in terms of x_t, rather than $\tilde{y}_t \equiv y_t - y_t^n$:

$$x_t = \mathbb{E}_t \left\{ x_{t+1} \right\} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t \left\{ \pi_{t+1} \right\} - \overline{r}_t^n \right), \quad \text{(IS)}$$

$$\pi_t = \beta \mathbb{E}_t \left\{ \pi_{t+1} \right\} + \kappa x_t + u_t, \quad \text{(PC)}$$

  where the exogenous terms are

$$\overline{r}_t^n \equiv r + \sigma \mathbb{E}_t \{ \Delta \overline{y}_{t+1}^n \} = r + \frac{\sigma(1+\varphi)}{\sigma(1-\alpha)+\varphi+\alpha} \mathbb{E}_t \{ \Delta a_{t+1} \},$$

$$u_t \equiv \kappa \left(\overline{y}_t^n - y_t^n \right) = \frac{1-\alpha}{\sigma(1-\alpha)+\varphi+\alpha} \left(\mu_t - \mu \right).$$

- We assume for simplicity that the cost-push shock u_t is i.i.d.

Rewriting the optimization problem I

- We proceed as if CB, at each date t,
  - directly controlled not only i_t, but also π_t and x_t,
  - observed the history of the exogenous shocks $(\overline{r}^n_{t-k}, u_{t-k})_{k\geq 0}$

  (these "working assumptions" will be relaxed in Chapter 3).

- So, loosely speaking, we are looking for the path of (π_t, x_t, i_t) minimizing L_t subject to (PC)-(IS).

- Note that L_t and (PC) do not involve i_t. So, choosing (π_t, x_t, i_t) to minimize L_t subject to (PC)-(IS) amounts to choosing (π_t, x_t) to minimize L_t subject to (PC) and determining residually i_t with (IS).

- In the following, we focus on the **reduced optimization problem** consisting in choosing (π_t, x_t) to minimize L_t subject to (PC).

Rewriting the optimization problem II

- So, loosely speaking, we are looking for the path of (x_t, π_t) minimizing

$$L_t \equiv \mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} \beta^{t+k} \left[\pi_{t+k}^2 + \lambda (x_{t+k} - x^*)^2 \right] \right\},$$

subject to $\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa x_t + u_t$.

- Note that in the absence of cost-push shocks ($u_t = 0$) and steady-state inefficiency ($x^* = 0$),
    - there is no **trade-off** between the π_t- and x_t-stabilization objectives,
    - as previously seen, optimal MP achieves the first best.

- We consider two alternative cases in turn:
    - **discretion**: at each date t, CB chooses (x_t, π_t) to minimize L_t,
    - **commitment**: at date 0, CB chooses $(x_t, \pi_t)_{t \geq 0}$ to minimize L_0.

Discretion I

- Under **discretion**, at each date t, CB chooses (x_t, π_t) to minimize

$$L_t \equiv \mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} \beta^{t+k} \left[\pi_{t+k}^2 + \lambda (x_{t+k} - x^*)^2 \right] \right\},$$

subject to $\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa x_t + u_t$, taking $\mathbb{E}_t \{\pi_{t+1}\}$ as given.

- The **first-order condition** (FOC) of this optimization problem is

$$\kappa \pi_t + \lambda x_t = \lambda x^*.$$

- Using (PC) to remove x_t from this FOC, we get

$$\mathbb{E}_t \{\pi_{t+1}\} = \left(\frac{\kappa^2 + \lambda}{\beta \lambda} \right) \pi_t - \left(\frac{\kappa}{\beta} \right) x^* - \left(\frac{1}{\beta} \right) u_t.$$

Discretion II

- Iterating forward and applying the $\mathbb{E}_t\left\{.\right\}$ operator yields

$$\begin{aligned}\mathbb{E}_t\left\{\pi_{t+k}\right\} &= \left(\frac{\kappa^2+\lambda}{\beta\lambda}\right)^k\left[\pi_t - \frac{\kappa\lambda}{\kappa^2+\lambda(1-\beta)}x^* - \frac{\lambda}{\kappa^2+\lambda}u_t\right] \\ &\quad + \frac{\kappa\lambda}{\kappa^2+\lambda(1-\beta)}x^*.\end{aligned}$$

- Since $(\kappa^2+\lambda)^2 > \beta\lambda^2$, the unique value of π_t consistent with a finite value of L_t is

$$\pi_t = \frac{\kappa\lambda}{\kappa^2+\lambda(1-\beta)}x^* + \frac{\lambda}{\kappa^2+\lambda}u_t.$$

- $\mathcal{P}^\star$ under **discretion** is characterized by the above expression for π_t and the following expression for x_t (obtained using the FOC):

$$x_t = \frac{\lambda(1-\beta)}{\kappa^2+\lambda(1-\beta)}x^* - \frac{\kappa}{\kappa^2+\lambda}u_t.$$

Commitment I

- Under **commitment at date** 0, CB chooses at date 0, once and for all, (x_t, π_t) as a function of $(u_{t-k})_{0 \leq k \leq t}$ for all $t \geq 0$ to minimize

$$L_0 \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\pi_t^2 + \lambda (x_t - x^*)^2 \right] \right\},$$

subject to $\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa x_t + u_t$ for all $t \geq 0$.

- Then, at each date $t \geq 0$, observing the shocks $(u_{t-k})_{0 \leq k \leq t}$, CB applies its date-0 decision.

- We follow the **method of undetermined coefficients**: without any loss in generality (given the linear-quadratic framework), we can write

$$\begin{aligned} \pi_t &= \sum_{k=0}^{t} a_k^\pi u_{t-k} + b_t^\pi, \\ x_t &= \sum_{k=0}^{t} a_k^x u_{t-k} + b_t^x. \end{aligned}$$

Commitment II

- The corresponding **Lagrangian** is

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\pi_t^2 + \lambda (x_t - x^*)^2 \right] \right.$$
$$\left. -2V_u \sum_{t=0}^{+\infty} \beta^t \chi_t \left(\pi_t - \beta \mathbb{E}_t \left\{ \pi_{t+1} \right\} - \kappa x_t - u_t \right) \right\},$$

where V_u denotes the variance of u_t.

- The six corresponding **first-order conditions** are

$$\begin{aligned}
a_0^\pi - \mathbb{E}_0 \left\{ \chi_t u_t \right\} &= 0, \\
a_k^\pi - \mathbb{E}_0 \left\{ \chi_t u_{t-k} \right\} + \mathbb{E}_0 \left\{ \chi_{t-1} u_{t-k} \right\} &= 0 \text{ for } k \geq 1, \\
\lambda a_k^x + \kappa \mathbb{E}_0 \left\{ \chi_t u_{t-k} \right\} &= 0 \text{ for } k \geq 0, \\
b_0^\pi - V_u \mathbb{E}_0 \left\{ \chi_0 \right\} &= 0, \\
b_t^\pi - V_u \mathbb{E}_0 \left\{ \chi_t \right\} + V_u \mathbb{E}_0 \left\{ \chi_{t-1} \right\} &= 0 \text{ for } t \geq 1, \\
\lambda (b_t^x - x^*) + V_u \kappa \mathbb{E}_0 \left\{ \chi_t \right\} &= 0 \text{ for } t \geq 0.
\end{aligned}$$

Commitment III

- (PC) gives three additional conditions:

$$\begin{aligned} \beta a_1^\pi - a_0^\pi + \kappa a_0^x &= -1, \\ \beta a_{k+1}^\pi - a_k^\pi + \kappa a_k^x &= 0 \text{ for } k \geq 1, \\ \beta b_{t+1}^\pi - b_t^\pi + \kappa b_t^x &= 0 \text{ for } t \geq 0. \end{aligned}$$

- The previous nine equations lead to the following four conditions on coefficients a^π and a^x:

$$\begin{aligned} \kappa a_0^\pi + \lambda a_0^x &= 0, \\ \beta a_1^\pi - a_0^\pi + \kappa a_0^x &= -1, \\ \kappa a_{k+1}^\pi + \lambda a_{k+1}^x - \lambda a_k^x &= 0 \text{ for } k \geq 0, \\ \beta a_{k+1}^\pi - a_k^\pi + \kappa a_k^x &= 0 \text{ for } k \geq 1, \end{aligned}$$

Commitment IV

- And they lead to the following three conditions on coefficients b^π and b^x:

$$\begin{aligned}
\kappa b_0^\pi + \lambda b_0^x &= \lambda x^*, \\
\kappa b_{t+1}^\pi + \lambda b_{t+1}^x - \lambda b_t^x &= 0 \text{ for } t \geq 0, \\
\beta b_{t+1}^\pi - b_t^\pi + \kappa b_t^x &= 0 \text{ for } t \geq 0.
\end{aligned}$$

- Coefficients b^π therefore satisfy the **recurrence equation** $\beta\lambda b_{t+2}^\pi - (\beta\lambda + \kappa^2 + \lambda) b_{t+1}^\pi + \lambda b_t^\pi = 0$ for $t \geq 0$, whose **characteristic polynomial** has two roots, which are both real numbers:

$$\begin{aligned}
\omega &\equiv \frac{(\beta\lambda + \kappa^2 + \lambda) - \sqrt{(\beta\lambda + \kappa^2 + \lambda)^2 - 4\beta\lambda^2}}{2\beta\lambda} \in (0, 1), \\
\omega' &\equiv \frac{(\beta\lambda + \kappa^2 + \lambda) + \sqrt{(\beta\lambda + \kappa^2 + \lambda)^2 - 4\beta\lambda^2}}{2\beta\lambda} > \beta^{-\frac{1}{2}} > 1.
\end{aligned}$$

Commitment V

- These coefficients are therefore of the form $b_t^\pi = \delta_b \omega^t + \delta_b' \omega'^t$ for $t \geq 0$.

- The two conditions pinning down δ_b and δ_b' are
  - the **initial condition** $\beta \lambda b_1^\pi - (\kappa^2 + \lambda) b_0^\pi = -\kappa \lambda x^*$,
  - the condition $\delta_b' = 0$, for L_0 to take a finite value (since $\beta \omega'^2 > 1$).

- We thus obtain

$$b_t^\pi = \frac{\lambda (1-\omega) \omega^t}{\kappa} x^*,$$

from which we get

$$b_t^x = \omega^{t+1} x^*,$$

for $t \geq 0$.

Commitment VI

- Similarly, coefficients a^{π} satisfy the **recurrence equation** $\beta\lambda a^{\pi}_{k+2} - (\beta\lambda + \kappa^2 + \lambda)\, a^{\pi}_{k+1} + \lambda a^{\pi}_k = 0$ for $k \geq 1$, so that they are of the form $a^{\pi}_k = \delta_a \omega^k + \delta'_a \omega'^k$ for $k \geq 1$.

- The three conditions pinning down δ_a, δ'_a and a^{π}_0 are
  - the **initial conditions** $\beta\lambda a^{\pi}_1 - (\kappa^2 + \lambda) a^{\pi}_0 = -\lambda$ and $\beta\lambda a^{\pi}_2 - (\beta\lambda + \kappa^2 + \lambda) a^{\pi}_1 + \lambda a^{\pi}_0 = \lambda$,
  - the condition $\delta'_a = 0$, for L_0 to take a finite value (since $\beta\omega'^2 > 1$).

- We thus obtain

$$
\begin{aligned}
a^{\pi}_0 &= \omega, \\
a^{\pi}_k &= -(1-\omega)\omega^k \text{ for } k \geq 1, \\
a^{x}_k &= -\frac{\kappa\omega^{k+1}}{\lambda} \text{ for } k \geq 0.
\end{aligned}
$$

Commitment VII

- Therefore, we eventually get $\mathcal{P}^{\star}$ under **commitment at date** 0:

$$\pi_t = \frac{\lambda(1-\omega)\omega^t}{\kappa}x^* + \omega u_t - (1-\omega)\sum_{k=1}^{t}\omega^k u_{t-k},$$
$$x_t = \omega^{t+1}x^* - \frac{\kappa\omega}{\lambda}\sum_{k=0}^{t}\omega^k u_{t-k},$$

- Under Woodford's (1999) "**timeless perspective**" (i.e. commitment at date $-\infty$), $\mathcal{P}^{\star}$ becomes

$$\pi_t = \omega u_t - (1-\omega)\sum_{k=1}^{+\infty}\omega^k u_{t-k},$$
$$x_t = -\frac{\kappa\omega}{\lambda}\sum_{k=0}^{+\infty}\omega^k u_{t-k}.$$

Inflation and stabilization biases

- The difference between $\mathcal{P}^\star$ under discretion and $\mathcal{P}^\star$ under commitment at date 0 is the result of both
  - an **inflation bias**, which arises when $x^* > 0$, i.e. when the steady-state output level is inefficiently low,
  - a **stabilization bias**, which arises when $u_t \neq 0$, i.e. when the economy is hit by cost-push shocks.

- Because these biases are independent of each other, we consider them separately in the rest of the chapter, and focus
  - first on the inflation bias alone, by assuming $x^* > 0$ and $u_t = 0$,
  - then on the stabilization bias alone, by assuming $x^* = 0$ and $u_t \neq 0$.

Inflation bias I

- In this section, we assume that $x^* > 0$ and $u_t = 0$.

- We then get, under **discretion** (denoted by superscript "d"),

$$\pi_t = \pi^d \equiv \frac{\kappa\lambda}{\kappa^2 + \lambda(1-\beta)} x^* \quad \text{and} \quad x_t = x^d \equiv \frac{\lambda(1-\beta)}{\kappa^2 + \lambda(1-\beta)} x^*,$$

under **commitment at date** 0 (denoted by superscript "c"),

$$\pi_t = \pi_t^c \equiv \frac{\lambda(1-\omega)\omega^t}{\kappa} x^* \quad \text{and} \quad x_t = x_t^c \equiv \omega^{t+1} x^*,$$

and under **timeless perspective** (denoted by superscript "tp"),

$$\pi_t = \pi^{tp} \equiv 0 \quad \text{and} \quad x_t = x^{tp} \equiv 0.$$

Inflation bias II

- Therefore, at each date $t \geq 0$, the inflation rate is higher under discretion than under commitment at date 0 and under timeless perspective:

$$\begin{aligned} \pi^d - \pi_t^c &\geq \frac{\beta\lambda^2(1-\omega)^2}{\kappa[\kappa^2+\lambda(1-\beta)]}x^* > 0, \\ \pi^d - \pi^{tp} &> 0, \end{aligned}$$

  hence the term "**inflation bias**".

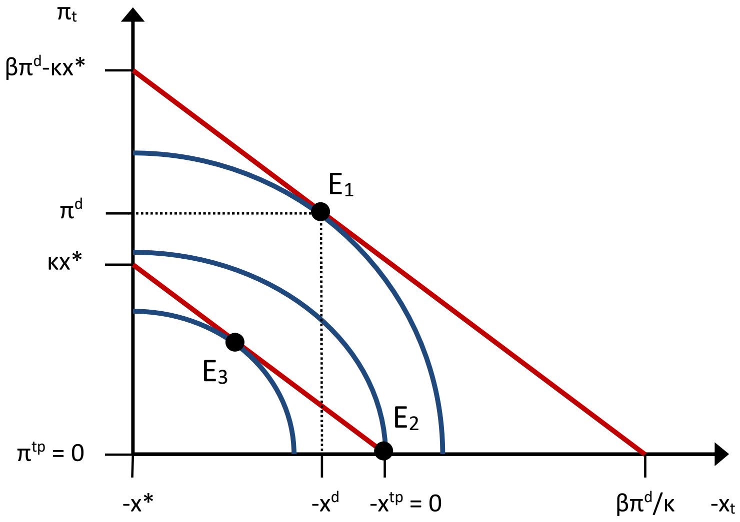
- Let L^d, L^c, and L^{tp} denote the value taken by $\mathbb{E}\{L_0\}$ respectively under discretion, commitment at date 0, and timeless perspective.

- We have $L^c < L^d$, $L^c < L^{tp}$, and $L^{tp} \lesseqgtr L^d$.

Interpretation of the inflation bias I

- Suppose that $L^{tp} < L^d$ (as is typically the case for standard calibrations).
- Consider **three alternative scenarios**:
  - Scenario 1: $\mathbb{E}_t\{\pi_{t+1}\} = \pi^d$, $x_t = x^d$, and $\pi_t = \pi^d$ (**discretion**),
  - Scenario 2: $\mathbb{E}_t\{\pi_{t+1}\} = 0$, $x_t = 0$, and $\pi_t = 0$ (**timeless persp.**),
  - Scenario 3: $\mathbb{E}_t\{\pi_{t+1}\} = 0$, $x_t = \frac{\lambda x^*}{\kappa^2+\lambda}$, and $\pi_t = \frac{\kappa\lambda x^*}{\kappa^2+\lambda}$.
- Scenarios 1, 2, and 3 correspond to Points E1, E2, and E3 respectively on the following graph, where the **Phillips curves** are drawn in red and the **iso-welfare curves** in blue.

Interpretation of the inflation bias II



Interpretation of the inflation bias III

- **Scenario 1** is **time-consistent**: if the private agents constantly expect $\mathbb{E}_t\{\pi_{t+1}\} = \pi^d$, then under discretion CB constantly chooses $\pi_t = \pi^d$.

- **Scenario 2** would be preferable to Scenario 1 (as it leads to a lower welfare loss) but is **time-inconsistent**.

- Indeed, if the private agents constantly expected Scenario 2 to happen, i.e. constantly expected $\mathbb{E}_t\{\pi_{t+1}\} = 0$, then under discretion CB would constantly choose $\pi_t = \frac{\kappa\lambda x^*}{\kappa^2+\lambda} \neq 0$, i.e. **Scenario 3** would constantly happen.

- If the private agents constantly expected Scenario 2 to happen, then under discretion CB would constantly depart from Scenario 2 by
  - increasing x_t in order to bring it closer to x^*, which provides initially a strictly positive marginal welfare gain,
  - increasing π_t, which has initially a zero marginal welfare cost,

  until the marginal welfare gain equals the marginal welfare cost, which happens at Scenario 3.

CB reputation

- **Reputation concerns** may eliminate the inflation bias by inducing CB to implement the timeless-perspective path.

- Assume that private agents follow a **grim-trigger strategy**: they expect the implementation of Scenario 2 as long as CB has not deviated from this scenario; if it deviates, then it looses its reputation for a given number of periods during which they expect the implementation of Scenario 1.

- Then, CB may have the incentive never to deviate from Scenario 2, as the medium-term cost due to the reputation loss would be higher than the short-term benefit.

- These reputation effects were first analyzed by Barro and Gordon (1983a, 1983b) in a neo-classical framework.

MP delegation

- If reputation concerns do not work, then one solution is to **delegate MP** to a CB whose preferences differ from society's, i.e. a CB that minimizes a loss function different from the welfare-loss function.

- This loss function should be such that the path minimizing it under discretion coincides with the path minimizing the welfare-loss function under commitment.

- Rogoff (1985) proposed to delegate MP to a "**conservative CB**", i.e. a CB whose preferences are described by the following instantaneous loss function: $(\pi_t)^2 + \lambda'(x_t - x^*)^2$, where $0 \leq \lambda' \leq \lambda$.

- If $\lambda' = 0$, then under discretion the CB implements Scenario 2.

- As can be seen on the next slide, all major CBs have been assigned **one main objective, price stability**, the only exception being the Fed.

MP objectives of major CBs

- **Bank of England**: price stability; "*conditionally on price stability, support to the economic policy of the government, including in its growth and employment objectives*."
- **Bank of Japan**: "*achieving* price stability, *thereby, contributing to the sound development of the national economy*."
- **European Central Bank**: price stability; "*without prejudice to the objective of price stability, support to the general economic policies in the European Community*."
- **Federal Reserve**: price stability; maximal employment; moderate long-term interest rates.

The Fed exception I

- Indeed, on January 25th, 2012, the Fed published the following statement:

  "The FOMC [Federal Open-Market Committee] is firmly committed to fulfilling its statutory mandate from the Congress of promoting maximum employment, stable prices, and moderate long-term interest rates. (...)

  The inflation rate over the longer run is primarily determined by monetary policy, and hence the Committee has the ability to specify a longer-run goal for inflation. The Committee judges that inflation at the rate of 2 percent, as measured by the annual change in the price index for personal consumption expenditures, is most consistent over the longer run with the Federal Reserve's statutory mandate. Communicating this inflation goal clearly to the public helps keep longer-term inflation expectations firmly anchored, thereby fostering price stability and moderate long-term interest rates and enhancing the Committee's ability to promote maximum employment in the face of significant economic disturbances.

The Fed exception II

The maximum level of employment is largely determined by nonmonetary factors that affect the structure and dynamics of the labor market. These factors may change over time and may not be directly measurable. Consequently, it would not be appropriate to specify a fixed goal for employment; rather, the Committee's policy decisions must be informed by assessments of the maximum level of employment, recognizing that such assessments are necessarily uncertain and subject to revision. (...)

In setting monetary policy, the Committee seeks to mitigate deviations of inflation from its longer-run goal and deviations of employment from the Committee's assessments of its maximum level. These objectives are generally complementary. However, under circumstances in which the Committee judges that the objectives are not complementary, it follows a balanced approach in promoting them, taking into account the magnitude of the deviations and the potentially different time horizons over which employment and inflation are projected to return to levels judged consistent with its mandate."

The Fed exception III

- If "maximum employment" refers to employment at the **undistorted steady state**, then the corresponding instantaneous loss function is $(\pi_t)^2 + \lambda'(x_t - x^*)^2$, which does not eliminate the inflation bias.

- If "maximum employment" refers to employment at the **distorted steady state**, then the corresponding instantaneous loss function is $(\pi_t)^2 + \lambda'(x_t)^2$, which eliminates the inflation bias.

- On August 27$^{\text{th}}$, 2020, the Fed published a new statement amending the 2012 statement in several ways.

- In particular, the new statement refers to "shortfalls" of employment from its maximum level, rather than "deviations" of employment from its maximum level as the previous statement.

The Fed exception IV

- The Fed explained this change as follows:

  "[P]rior to the pandemic we witnessed a record long expansion in which the labor market was very strong and did not trigger a significant rise in inflation. The gains in the labor market were widely shared across society, and thus the revised statement reflects a greater appreciation that the benefits of a strong labor market may be sustained without triggering an unwelcome rise in inflation. (...)

  Accordingly, the new Statement on Longer-Run Goals and Monetary Policy strategy only refers to 'shortfalls of employment from its maximum level' rather than the 'deviations from its maximum level' used in the previous statement. This change signals that high employment, in the absence of unwanted increases in inflation (...), will not by itself be a cause for policy concern."

Illustration of the inflation bias: Volcker's disinflation I

Annual inflation rate in the United States (1961-2009)

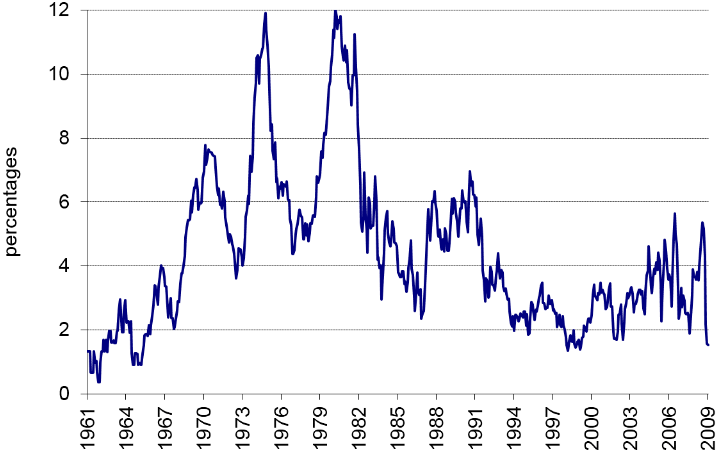


Illustration of the inflation bias: Volcker's disinflation II

Fed Funds rate (1979-1985)

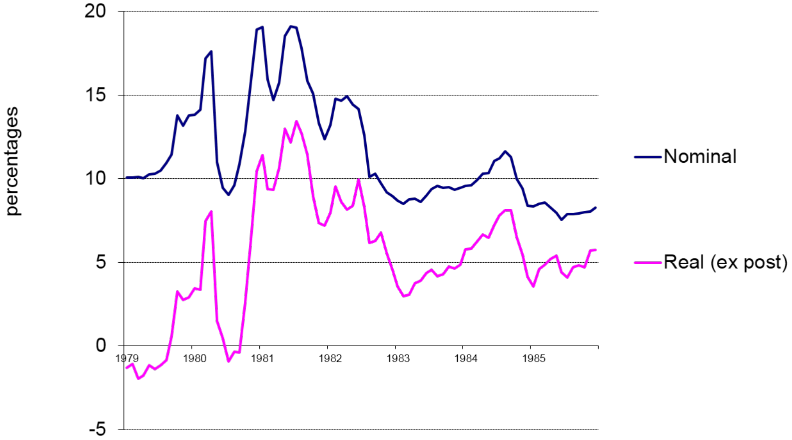
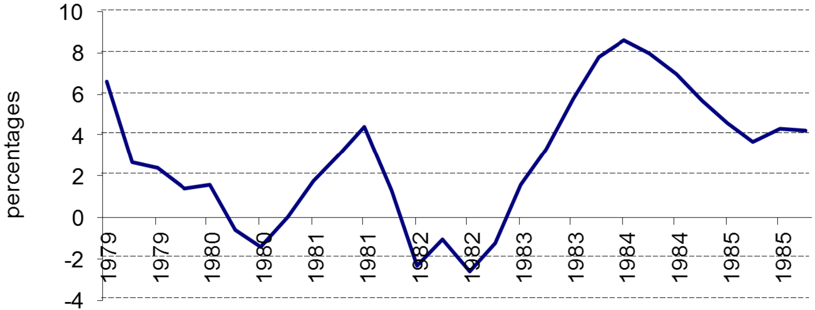


Illustration of the inflation bias: Volcker's disinflation III

Annual real GDP growth rate in the US (1979-1985)



Stabilization bias I

- In this section, we assume that $x^* = 0$ and $u_t \neq 0$.

- We then get, under **discretion** (denoted by superscript "d"):

$$\pi_t = \pi_t^d \equiv \frac{\lambda}{\kappa^2 + \lambda} u_t,$$
$$x_t = x_t^d \equiv \frac{-\kappa}{\kappa^2 + \lambda} u_t,$$

and under **commitment at date** 0 (denoted by superscript "c"):

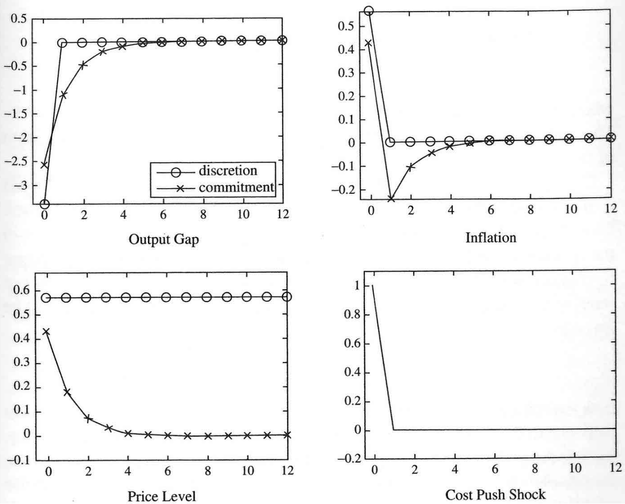
$$\pi_t = \pi_t^c \equiv \omega u_t - (1 - \omega) \sum_{k=1}^{t} \omega^k u_{t-k},$$
$$x_t = x_t^c \equiv -\frac{\kappa\omega}{\lambda} \sum_{k=0}^{t} \omega^k u_{t-k}.$$

Stabilization bias II

- The two paths are different: there is a "**stabilization bias**" under discretion (Clarida et al., 1999; Woodford, 1999).
- In particular, the effects of a shock u_t are **persistent** under commitment, unlike under discretion.
- Under commitment, inflation reacts **positively** to a cost-push shock **on impact** ($\partial \pi_t / \partial u_t > 0$), and **negatively afterwards** ($\partial \pi_{t+k} / \partial u_t < 0$ for $k \geq 1$).
- Under commitment, **the price level is stationary** (i.e. it eventually goes back to its initial value, following a one-off cost-push shock):

$$\frac{\partial \left(p_{+\infty} - p_{t-1} \right)}{\partial u_t} = \frac{\partial \sum_{k=0}^{+\infty} \pi_{t+k}}{\partial u_t} = \omega - (1 - \omega) \sum_{k=1}^{+\infty} \omega^k = 0.$$

Response to a one-off cost-push shock u_0



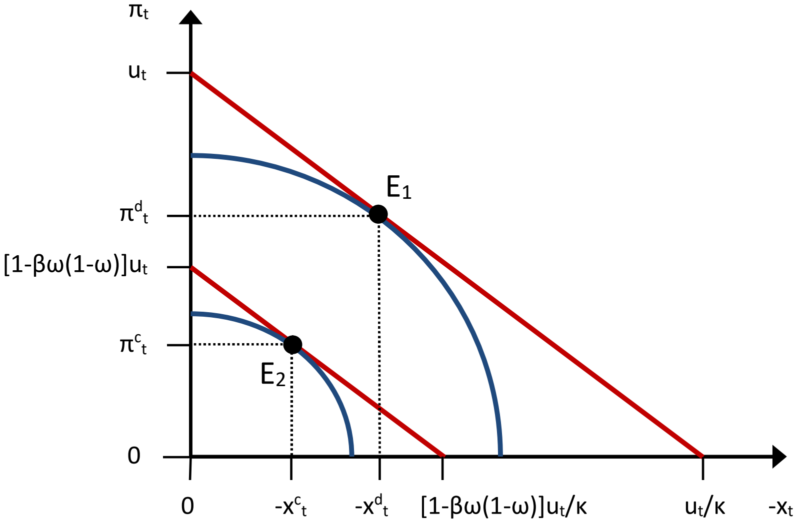
Source: Galí (2015, C5).

Interpretation of the stabilization bias I

- Consider **two alternative scenarios**:
  - Scenario 1: $\mathbb{E}_t\{\pi_{t+1}\} = 0$, $\pi_t = \pi_t^d$ and $x_t = x_t^d$ (**discretion**),
  - Scenario 2: $\mathbb{E}_t\{\pi_{t+1}\} = -(1-\omega)\omega u_t$, $\pi_t = \pi_t^c$ and $x_t = x_t^c$ (**commitment**).

- Scenarios 1 and 2 correspond to Points E1 and E2 respectively on the following graph, where the **Phillips curves** are drawn in red and the **iso-welfare curves** in blue.

Interpretation of the stabilization bias II



Interpretation of the stabilization bias III

- **Scenario 1** is **time-consistent**: if the private agents set $\mathbb{E}_t\{\pi_{t+1}\} = 0$, then under discretion CB chooses $\pi_t = \pi_t^d$.
- **Scenario 2** would be preferable to Scenario 1 (as it leads to a lower welfare loss) but is **time-inconsistent**.
- Indeed, under discretion, at date $t+1$, CB chooses a value for π_{t+1} that is independent of u_t, since u_t has stopped hitting the economy.

Interpretation of the stabilization bias IV

- The **optimal MP under commitment** (Scenario 2) consists in spreading over time the burden of adjustment to cost-push shocks, i.e. in making the short-term nominal interest rate react to these shocks in a gradual way and longer than they last.

- This reaction enables CB to have a sizeable initial effect on the long-term nominal interest rate and, therefore, on the output gap and the inflation rate.

- The **optimal MP under discretion** (Scenario 1) consists in making the short-term nominal interest rate react only when the shock hits, in a more aggressive way than the optimal MP under commitment.

CB reputation and communication

- Again, **reputation concerns** may eliminate the stabilization bias by inducing CB to implement the timeless-perspective path.
- Assume that private agents follow a **grim-trigger strategy**: they expect the implementation of Scenario 2 as long as CB has not deviated from this scenario; if it deviates, then it looses its reputation for a given number of periods during which they expect the implementation of Scenario 1.
- Then, CB may have the incentive never to deviate from Scenario 2, as the medium-term cost due to the reputation loss would be higher than the short-term benefit.
- One way for CB to commit in advance through reputation concerns is to **announce publicly** its policy objectives and plans, in order to overcome the inflation and stabilization biases.
- This role of CB communication is discussed by Bernanke (2003a).

In Bernanke's (2003a) words I

"Why have inflation-targeting central banks emphasized communication, particularly the communication of policy objectives, policy framework, and economic forecasts? In the 1960s, many economists were greatly interested in adapting sophisticated mathematical techniques developed by engineers for controlling missiles and rockets to the problem of controlling the economy. At the time, this adaptation of so-called stochastic optimal control methods to economic policymaking seemed natural; for like a ballistic missile, an economy may be viewed as a complicated dynamic system that must be kept on course, despite continuous buffeting by unpredictable forces.

Unfortunately, macroeconomic policy turned out not to be rocket science! The problem lay in a crucial difference between a missile and an economy–which is that, unlike the people who make up an economy, the components of a missile do not try to understand and anticipate the forces being applied to them. Hence, although a given propulsive force always has the same, predictable effect on a ballistic missile, a given policy action–say, a 25-basis-point cut in the federal funds rate–can have very different effects on the economy, depending (for...

In Bernanke's (2003a) words II

...example) on what the private sector infers from that action about likely future policy actions, about the information that may have induced the policymaker to act, about the policymaker's objectives in taking the action, and so on. Thus, taking the "right" policy action–in this case, changing the federal funds rate by the right amount at the right time–is a necessary but not sufficient condition for getting the desired economic response.

Most inflation-targeting central banks have found that effective communication policies are a useful way, in effect, to make the private sector a partner in the policymaking process. To the extent that it can explain its general approach, clarify its plans and objectives, and provide its assessment of the likely evolution of the economy, the central bank should be able to reduce uncertainty, focus and stabilize private-sector expectations, and–with intelligence, luck, and persistence–develop public support for its approach to policymaking. Of course, as has often been pointed out, actions speak louder than words; and declarations by the central bank will have modest and diminishing value if they are not clear, coherent, and–most important–credible, in the sense of being consistently backed up by action. (...)

In Bernanke's (2003a) words III

One objection that has been raised to the public announcement of policy objectives, economic forecasts, and (implicit or explicit) policy plans by central banks is that even relatively modest commitments along these lines may limit their flexibility to choose the best policies in the future. Isn't it always better to be more rather than less flexible? Shouldn't the considered judgment of experienced policymakers always trump rules, even relatively flexible ones, for setting policy?

I agree that human judgment should always be the ultimate source of policy decisions and that no central bank should–or is even able to–commit irrevocably in advance to actions that may turn out to be highly undesirable. However, the intuition that more flexibility is always better than less flexibility is quite fallacious, a point understood long ago by Homer, who told of how Ulysses had himself tied to the mast so as not to fall victim to the songs of the Sirens. More recently, the notion that more flexibility is always preferable has been pretty well gutted by modern game theory (not to mention modern monetary economics), which has shown in many contexts that the ability to commit in advance often yields better outcomes."

MP delegation I

- Again, if reputation concerns do not work, then one solution is to **delegate MP**, i.e. to assign to CB the objective of minimizing a loss function that differs from the welfare-loss function.

- This loss function should be such that the path minimizing it under discretion coincides with the path minimizing the welfare-loss function under commitment.

- The literature has proposed a number of such **MP delegation schemes**, summarized in the following table.

- These schemes make MP **inertial** like the optimal MP under commitment (Scenario 2), in the sense that MP reacts in a persistent way to a one-off cost-push shock, in order to smooth the effect of the shock over time.

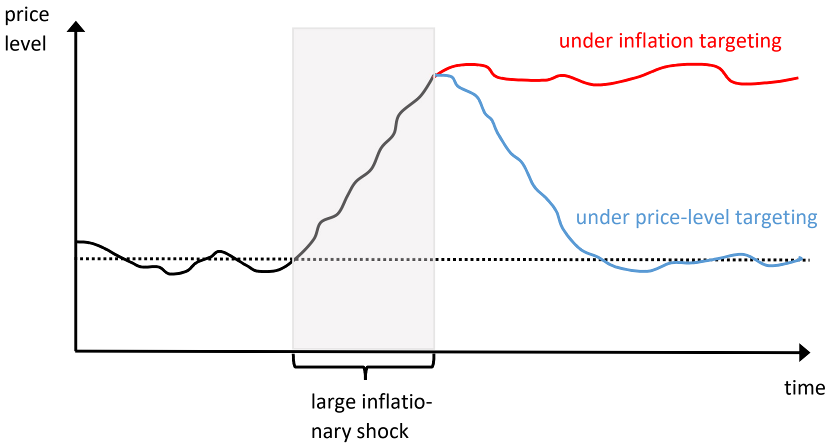
MP delegation II

Target	**Instantaneous loss function**	**Paper**
Price level	$(p_t)^2 + \lambda'(x_t)^2$	Vestin (2006)
Nominal-output growth	$(\pi_t)^2 + \lambda'(x_t)^2 + d(\pi_t + \Delta y_t)^2$	Jensen (2002)
Output-gap change	$(\pi_t)^2 + \lambda'(\Delta x_t)^2$	Walsh (2003)
Inflation-expectation change	$(\pi_t)^2 + \lambda'(x_t)^2 +$ $d(\mathbb{E}_t\{\pi_{t+1}\} - \mathbb{E}_{t-1}\{\pi_{t+1}\})$	Svensson and Woodford (2005)

(In all cases, $\lambda' > 0$ and $d > 0$.)

Price-level path under inflation vs. price-level targeting

(for a zero inflation target vs. a constant price-level target)



The Fed's average-inflation-stabilization objective

- On August 27th, 2020, the Fed announced, in a statement, the adoption of a **average-inflation-stabilization objective**, which is close to a **price-level-stabilization objective**:

  "*[T]he Committee seeks to achieve inflation that averages 2 percent over time, and therefore judges that, following periods when inflation has been running persistently below 2 percent, appropriate monetary policy will likely aim to achieve inflation moderately above 2 percent for some time.*"

- As the Fed explained in an accompanying document, however, its motivation for adopting this objective had to do with the zero-lower-bound (ZLB) constraint on nominal interest rates, rather than with cost-push shocks.

- So, we will analyze this development in Chapter 4, when we study monetary policy at the ZLB.

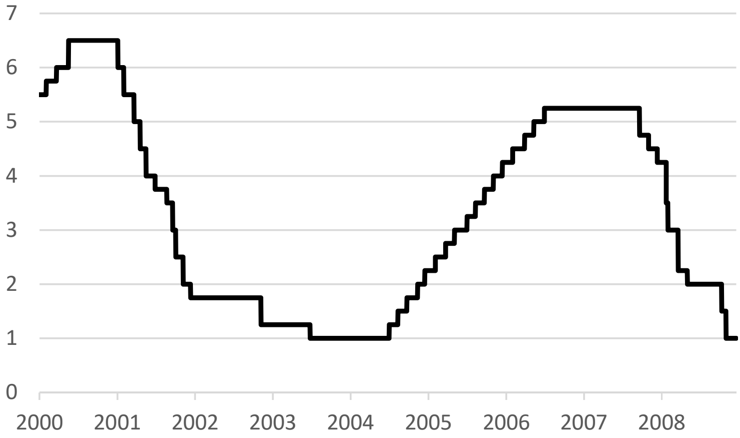
MP gradualism

- The optimal MP under commitment is **inertial**, in the sense that it makes the short-term nominal interest rate react to cost-push shocks in an inertial way.
- This policy is related to the **gradualism** adopted by many CBs and consisting in moving policy rates by small steps going in the same direction (Woodford, 2003b).
- This gradualism increases the **predictability** of future moves in the short-term nominal interest rate and, therefore, CB's capacity to affect the long-term nominal interest rate and, via the latter, the output gap and the inflation rate.
- In Bernanke's (2004b) words: "*by leading market participants to anticipate that changes in the policy rate will be followed by further changes in the same direction, policy gradualism may increase the ability of the Fed to affect long-term rates and thus influence economic behavior.*"

Gradualism of the Fed's monetary policy I

Federal Funds target rate, January 2000 – December 2008

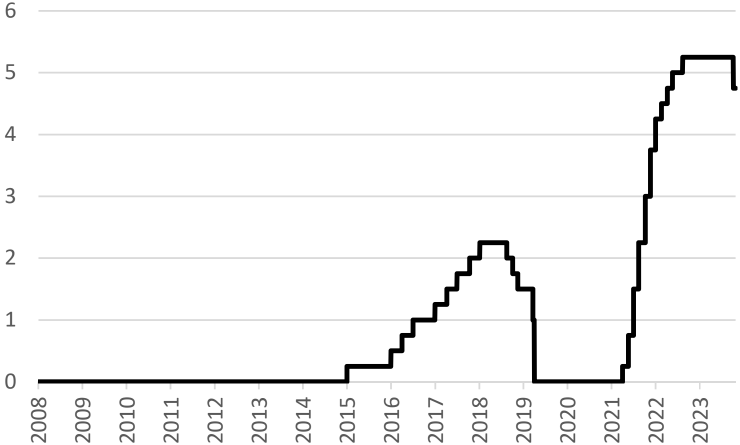
(in percentage points per year)



Gradualism of the Fed's monetary policy II

Lower limit of Federal Funds target range, December 2008 – October 2024

(in percentage points per year)



Determination of the welfare-loss function I

- For any variable Z_t, we have

$$\frac{Z_t - Z}{Z} \simeq \widehat{z}_t + \frac{\widehat{z}_t^2}{2},$$

where $\widehat{z}_t \equiv z_t - z$ is the log-deviation of Z_t from its ZIRSS value.

- Therefore, using the market-clearing condition $\widehat{c}_t = \widehat{y}_t$, we get

$$U_t - U \simeq U_c C \left(\frac{C_t - C}{C} \right) + U_n N \left(\frac{N_t - N}{N} \right) + \frac{U_{cc} C^2}{2} \left(\frac{C_t - C}{C} \right)^2$$
$$+ \frac{U_{nn} N^2}{2} \left(\frac{N_t - N}{N} \right)^2 \simeq U_c C \left(\widehat{y}_t + \frac{1-\sigma}{2} \widehat{y}_t^2 \right) + U_n N \left(\widehat{n}_t + \frac{1+\varphi}{2} \widehat{n}_t^2 \right).$$

Determination of the welfare-loss function II

- Recall from Chapter 1 that

$$\widehat{y}_t = (1-\alpha)\widehat{n}_t + a_t - d_t,$$

where $d_t \equiv (1-\alpha)\log\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{\frac{-\varepsilon_t}{1-\alpha}} di$.

- **Lemma 1**: *up to a second-order approximation,* $d_t \simeq \frac{\varepsilon}{2\Theta} var_i\{p_t(i)\}$, *where* $\Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$.

- To show Lemma 1, use first the definition of the price index P_t to get

$$\begin{aligned} 1 &= \int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{1-\varepsilon_t} di = \int_0^1 \exp\{(1-\varepsilon_t)[p_t(i)-p_t]\} di \\ &\simeq 1 + (1-\varepsilon_t)\int_0^1 [p_t(i)-p_t] di + \frac{(1-\varepsilon_t)^2}{2}\int_0^1 [p_t(i)-p_t]^2 di. \end{aligned}$$

Determination of the welfare-loss function III

- Therefore, up to second order,

$$\begin{aligned}\int_0^1 [p_t(i) - p_t] di &\simeq -\left(\frac{1-\varepsilon_t}{2}\right)\int_0^1 [p_t(i) - p_t]^2 di \\ &\simeq -\left(\frac{1-\varepsilon}{2}\right)\int_0^1 [p_t(i) - p_t]^2 di.\end{aligned}$$

- In addition,

$$\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\frac{\varepsilon_t}{1-\alpha}} di = \int_0^1 \exp\left\{-\frac{\varepsilon_t}{1-\alpha}[p_t(i) - p_t]\right\} di$$

$$\simeq \quad 1 - \frac{\varepsilon_t}{1-\alpha}\int_0^1 [p_t(i) - p_t] di + \frac{1}{2}\left(\frac{\varepsilon_t}{1-\alpha}\right)^2 \int_0^1 [p_t(i) - p_t]^2 di$$

$$\simeq \quad 1 + \frac{1}{2}\left[\frac{\varepsilon_t(1-\varepsilon)}{1-\alpha}\right]\int_0^1 [p_t(i) - p_t]^2 di + \frac{1}{2}\left(\frac{\varepsilon}{1-\alpha}\right)^2 \int_0^1 [p_t(i) - p_t]^2 di$$

Determination of the welfare-loss function IV

$$\begin{aligned}
&\simeq && 1+\frac{1}{2}\left[\frac{\varepsilon(1-\varepsilon)}{1-\alpha}\right]\int_0^1 [p_t(i)-p_t]^2 di + \frac{1}{2}\left(\frac{\varepsilon}{1-\alpha}\right)^2 \int_0^1 [p_t(i)-p_t]^2 di \\
&= && 1+\frac{1}{2}\left(\frac{\varepsilon}{1-\alpha}\right)\frac{1}{\Theta}\int_0^1 [p_t(i)-p_t]^2 di \\
&\simeq && 1+\frac{1}{2}\left(\frac{\varepsilon}{1-\alpha}\right)\frac{1}{\Theta} var_i\{p_t(i)\},
\end{aligned}$$

where the last equality follows from the fact that, up to second order,

$$var_i\{p_t(i)\} \equiv \int_0^1 \left[p_t(i) - \int_0^1 p_t(i)di\right]^2 di \simeq \int_0^1 [p_t(i)-p_t]^2 di.$$

- Therefore, $d_t \simeq \frac{\varepsilon}{2\Theta} var_i\{p_t(i)\}$, which proves Lemma 1.

Determination of the welfare-loss function V

- We can then rewrite $U_t - U$ as

$$U_t - U \simeq U_c C \left(\widehat{y}_t + \frac{1-\sigma}{2} \widehat{y}_t^2 \right)$$
$$+ \frac{U_n N}{1-\alpha} \left[\widehat{y}_t + \frac{\varepsilon}{2\Theta} var_i \{ p_t(i) \} + \frac{1+\varphi}{2(1-\alpha)} (\widehat{y}_t - a_t)^2 \right] + t.i.p.,$$

where $t.i.p.$ stands for "terms independent of policy."

- Let Φ denote the size of the steady-state inefficiency, implicitly defined by $-\frac{U_n}{U_c} = MPN(1-\Phi)$ and assumed to be "small" (i.e. a first-order term).

Determination of the welfare-loss function VI

- Using $MPN = (1-\alpha)\frac{Y}{N}$, we get $\frac{U_t - U}{U_c C} \simeq$

$$\widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - (1-\Phi)\left[\widehat{y}_t + \frac{\varepsilon}{2\Theta} var_i\{p_t(i)\} + \frac{1+\varphi}{2(1-\alpha)}(\widehat{y}_t - a_t)^2\right] + t.i.p.$$

$$\begin{aligned}
&\simeq \quad \Phi\widehat{y}_t - \frac{1}{2}\left[\frac{\varepsilon}{\Theta} var_i\{p_t(i)\} - (1-\sigma)\widehat{y}_t^2 + \frac{1+\varphi}{1-\alpha}(\widehat{y}_t - a_t)^2\right] + t.i.p. \\
&= \quad \Phi\widehat{y}_t - \frac{1}{2}\left[\frac{\varepsilon}{\Theta} var_i\{p_t(i)\} + \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\widehat{y}_t^2 - 2\frac{1+\varphi}{1-\alpha}\widehat{y}_t a_t\right] + t.i.p. \\
&= \quad \Phi\widehat{y}_t - \frac{1}{2}\left[\frac{\varepsilon}{\Theta} var_i\{p_t(i)\} + \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)(\widehat{y}_t^2 - 2\widehat{y}_t\widehat{y}_t^e)\right] + t.i.p. \\
&= \quad \Phi x_t - \frac{1}{2}\left[\frac{\varepsilon}{\Theta} var_i\{p_t(i)\} + \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)x_t^2\right] + t.i.p.,
\end{aligned}$$

where we have used $\widehat{y}_t^e \equiv y_t^e - y^e = \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}a_t$ and $x_t \equiv y_t - \overline{y}_t^n$ $= y_t - (y_t^e - y^e + y) = \widehat{y}_t - \widehat{y}_t^e$.

Determination of the welfare-loss function VII

- Now, up to first order, we have

$$-\frac{U_n}{U_c} = MPN(1-\Phi) \quad \Longrightarrow \quad N^{\varphi} C^{\sigma} = (1-\alpha)\frac{Y}{N}(1-\Phi)$$
$$\Longrightarrow \quad N^{1+\varphi} Y^{\sigma-1} = (1-\alpha)(1-\Phi)$$
$$\Longrightarrow \quad \Phi \simeq \log(1-\alpha) + (1-\sigma)y - (1+\varphi)n$$
$$\Longrightarrow \quad \Phi \simeq \log(1-\alpha) + \frac{1+\varphi}{1-\alpha}a - \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y.$$

- Similarly, $0 = \log(1-\alpha) + \frac{1+\varphi}{1-\alpha}a - \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)y^e$, so that we get

$$\Phi \simeq \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)(y^e - y) = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)x^*.$$

- Therefore,

$$\frac{U_t - U}{U_c C} \simeq -\frac{1}{2}\left[\frac{\varepsilon}{\Theta} var_i\{p_t(i)\} + \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)(x_t - x^*)^2\right] + t.i.p.$$

Determination of the welfare-loss function VIII

- **Lemma 2**: $\sum_{t=0}^{+\infty} \beta^t var_i\{p_t(i)\} \simeq \frac{\theta}{(1-\beta\theta)(1-\theta)} \sum_{t=0}^{+\infty} \beta^t \pi_t^2$.

- Proof of Lemma 2: see Woodford (2003a, C6).

- Therefore, using $\chi \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$, we get $\mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left(\frac{U_t - U}{U_c C} \right) \right\} \simeq$

$$-\frac{1}{2}\mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\frac{\varepsilon}{\chi}\pi_t^2 + \left(\sigma + \frac{\varphi + \alpha}{1 - \alpha} \right) (x_t - x^*)^2 \right] \right\} + t.i.p.$$

- Hence the **welfare-loss function**

$$L_0 \equiv \mathbb{E}_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \left[\pi_t^2 + \lambda (x_t - x^*)^2 \right] \right\},$$

where $\lambda \equiv \left(\sigma + \frac{\varphi+\alpha}{1-\alpha} \right) \frac{\chi}{\varepsilon} = \frac{\kappa}{\varepsilon}$.